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AN APPROXIMATE TREATMENT OF A HEAT CONDUCTION PROBLEM INVOLVING A TWO-DIMENSIONAL SOLIDIFICATION FRONT

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Abstract—The approximate integral-methods of boundary layer theory in fluid dynamics are used to obtain information on a heat conduction problem involving a moving two-dimensional solidification front. The example treated is the inward solidification of a uniform prism having square cross-section and filled with liquid initially at the fusion temperature. The results obtained for the location and time history of the front are in good agreement with those found by Allen and Severn [1] using the relaxation method.

1. INTRODUCTION

THE problem discussed in this paper is that of the inward solidification of a liquid, initially at fusion temperature, contained in a uniform prism of square cross-section whose surface is maintained at a constant temperature below the fusion temperature of the liquid. The location and time history of the moving two-dimensional solidification front has been found, for a particular liquid, by Allen and Severn [1] using the relaxation method.

This problem has as yet proved intractable by exact analysis and it is the purpose of the present paper to give an approximate treatment, which is based on the approximate integral-methods for solving the boundary layer equations in fluid dynamics. These methods, namely the Kármán–Pohlhausen method [2] and the Tani method [3], reduce the mathematical problem of finding the two-dimensional front to the numerical integration of an ordinary first-order differential equation. Similar techniques have been applied in [4] to various problems involving one-dimensional solidification fronts. In a previous paper by Goodman [5] the Kármán–Pohlhausen method has been used to discuss the solidification of a semi-infinite region of fluid.

2. THE EQUATIONS OF THE PROBLEM

In the following discussion the liquid is assumed to satisfy the conditions: (i) the liquid has a definite fusion temperature, (ii) initially the liquid is at fusion temperature and (iii) all thermal properties of the material are uniform and constant. We consider a uniform prism of material having square cross-section and bounded by the isothermal surface

$$C_0(x, y) = (x^2 - a^2)(y^2 - a^2) = 0$$

at temperature T_0, where $T_0 < T_F$, the fusion temperature of the liquid. At any time during the period of inward solidification let the location of the two-dimensional solidification front be given by the surface $C_F(x, y, t) = 0$, an isothermal on which $T = T_F$. In accordance with the previous assumptions the equations describing the inward solidification process are:

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}, \qquad (2.1)$$

subject to the boundary conditions:

$$T = T_0 \text{ on } C_0(x, y) = 0 \text{ for } t > 0, \qquad (2.1a)$$

$$T = T_F \text{ on } C_F(x, y, t) = 0,$$

and at

$$t = 0,\ T = T_F \text{ and } C_F = C_0 = 0. \qquad (2.1b)$$

In (2.1) k denotes the thermal diffusivity. A further condition at the solid–liquid interface is required to determine the location and time history of the solidification front. Let $V(C_F, C_0)$

be the volume of solidified material at time t per unit depth of boundary in the axial direction. Thus in time δt the volume of solid increases by an amount $\delta V(C_F, C_0)$ and there will be set free an amount of heat

$$\delta Q = \rho L \, \delta V(C_F, C_0), \tag{2.2}$$

where ρ is the density and L the latent heat of fusion of the material. This must escape by conduction through the solidified material such that the amount of heat that flows outward is

$$\delta Q = -\kappa \left(\oint \frac{\partial T}{\partial \nu} \, ds \right) \delta t, \tag{2.3}$$

where κ is the thermal conductivity and s is the distance measured along the contour $C_F(x, y, t) = 0$ in an anticlockwise direction. Thus in the limit as δt becomes vanishingly small

$$\kappa \oint_{C_F = 0} \frac{\partial T}{\partial \nu} = -\rho L \frac{dV(C_F, C_0)}{dt}, \tag{2.4}$$

where

$$V(C_F, C_0) = \iint_{C_F = 0}^{C_0 = 0} dx \, dy = \iint_{C_F = 0}^{C_0 = 0} dS. \tag{2.5}$$

Introducing the dimensionless moduli:

$$\left.\begin{aligned} &\Theta = \frac{(T - T_0)}{(T_F - T_0)}, \quad \beta = \frac{\rho L k}{\kappa (T_F - T_0)}, \\ &\tau = \frac{tk}{a^2}, \quad v = \frac{V}{a^2}, \\ &l = \frac{s}{a}, \quad \sigma = \frac{S}{a^2}, \\ &X = \frac{x}{a} \text{ and } Y = \frac{y}{a}, \end{aligned}\right\} \tag{2.6}$$

the above equations (2.1) and (2.4) may be written in the dimensionless form:

$$\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} = \frac{\partial \Theta}{\partial \tau}; \tag{2.7}$$

$$\left.\begin{aligned} &\Theta = 0 \text{ on } C_0(X, Y) \\ &\quad = (X^2 - 1)(Y^2 - 1) = 0 \text{ for } \tau > 0, \\ &\Theta = 1 \text{ on } C_F(X, Y, \tau) = 0; \end{aligned}\right\} \tag{2.7a}$$

also

$$\oint_{C_F = 0} \frac{\partial \Theta}{\partial \nu} \, dl = -\beta \frac{dv}{d\tau}, \tag{2.7b}$$

where

$$v(C_F, C_0) = \iint_{C_F = 0}^{C_0 = 0} d\sigma.$$

The initial condition is that at

$$\tau = 0, \quad \Theta = 1 \tag{2.7c}$$

implying that solidification has not yet commenced.

3. APPROXIMATE ANALYSIS FOR EQUATIONS (2.7)

In order to apply the Kármán–Pohlhausen method or the Tani method to obtain approximate solutions of equations (2.7) we begin by assuming the shape or mathematical form of the solidification front, $C_F(X, Y, \tau) = 0$. We note that when

$$\tau = 0, \; C_F = C_0 = (X^2 - 1)(Y^2 - 1) = 0.$$

Intuitively we can argue that for small time an interface contour will be in the shape of a square having rounded corners and must become circular, i.e. of the form $X^2 + Y^2 - f(\tau) = 0$, for times near the finish of the solidification period. Moreover at the end of the solidification period the solidification front lies on the axis of the prism, $X = Y = 0$. It is thus reasonable to assume the following shape

$$C_F(X, Y, \tau) = (X^2 - 1)(Y^2 - 1) - \epsilon(\tau) = 0, \tag{3.1}$$

where ϵ is an unknown function of the time τ. The initial condition (2.7c) is now replaced by

$$\epsilon = 0 \text{ at } \tau = 0, \tag{3.2}$$

and by virtue of (3.1) $\epsilon = 1$ at the instant of complete solidification of the prism.

We now obtain certain integrals of the heat conduction equation (2.7) for the solidified phase. The heat balance integral, analogous to the momentum integral in boundary layer theory, is derived by integrating both sides of

(2.7) over the solidified phase bounded by the contours $C_0 = 0$ and $C_F = 0$. Applying Green's theorem to the resulting equation and using boundary condition (2.7b) we obtain:

$$\beta \frac{\mathrm{d}v}{\mathrm{d}\tau} + \oint_{C_0=0} \frac{\partial \Theta}{\partial \nu}\, \mathrm{d}l = \iint_{C_F=0}^{C_0=0} \frac{\partial \Theta}{\partial \tau}\, \mathrm{d}\sigma. \quad (3.3)$$

Furthermore on using the divergence theorem

$$\operatorname{div}(\Theta \nabla \Theta) = \nabla \Theta \,.\, \nabla \Theta + \Theta \nabla^2 \Theta,$$

we obtain for the solidified phase

$$\operatorname{div}(\Theta \nabla \Theta) = \nabla \Theta \,.\, \nabla \Theta + \Theta \frac{\partial \Theta}{\partial \tau}.$$

On applying Green's theorem and using the boundary conditions (2.7a) and (2.7b) we obtain a second integral, namely

$$\beta \frac{\mathrm{d}v}{\mathrm{d}\tau} = \iint_{C_F=0}^{C_0=0} \left\{ \left(\frac{\partial \Theta}{\partial X}\right)^2 + \left(\frac{\partial \Theta}{\partial Y}\right)^2 + \Theta \frac{\partial \Theta}{\partial \tau} \right\} \mathrm{d}\sigma.$$

The two integrals (3.3) and (3.4), as in the approximate methods of boundary layer theory, may now be used to find ϵ.

I.—*The Kármán–Pohlhausen method*

We assume for the solidified phase the one-parameter temperature distribution

$$\Theta = \frac{(X^2 - 1)(Y^2 - 1)}{\epsilon}, \quad (3.5)$$

which satisfies, by virtue of expression (3.1), the boundary conditions (2.7a). On substituting expression (3.5) into the heat balance integral (3.3) there results a first order differential equation for $\epsilon(\tau)$ to be solved subject to the boundary condition (3.2). On solving we obtain

$$\tau = \int_0^{\epsilon} \{\tfrac{3}{8}\beta A_0(\epsilon) + A_1(\epsilon)\}\, \mathrm{d}\epsilon, \quad (3.6)$$

where

$$A_0 = -\epsilon \frac{\mathrm{d}I_0}{\mathrm{d}\epsilon} \quad (3.7)$$

Table 1

ϵ	A_0	A_1	A_2	A_3	A_4	B_0	B_1	B_2
0·00	0·000000	0·000000	0·000000	0·000000	0·000000	2·666667	5·333333	3·555556
0·04	0·060322	0·012226	0·015893	0·047067	0·024866	2·625886	5·224356	3·473722
0·08	0·107342	0·021931	0·028433	0·084094	0·044463	2·583899	5·111826	3·389078
0·12	0·149558	0·030715	0·039754	0·117474	0·062084	2·540902	4·996325	3·302086
0·16	0·188741	0·038914	0·050300	0·148541	0·078471	2·496993$_5$	4·878149	3·212981
0·20	0·225720$_5$	0·046685	0·060282	0·177925	0·093964	2·452240$_5$	4·757497$_5$	3·121924
0·24	0·260979	0·054122	0·069822$_5$	0·205991	0·108756	2·406694	4·634522	3·029034
0·28	0·294833	0·061284	0·079001	0·232979	0·122976	2·360394	4·509341	2·934407
0·32	0·327503	0·068213	0·087874	0·259057	0·136713	2·313373	4·382054	2·838121
0·36	0·359154$_5$	0·074943	0·096483	0·284350	0·150034	2·265660	4·252744$_5$	2·740242
0·40	0·389913$_5$	0·081496	0·104861	0·308955	0·162989	2·217280	4·121485	2·640828
0·44	0·419880	0·087893	0·113033	0·332948	0·175620	2·168253	3·988339	2·539929
0·48	0·449136	0·094149	0·121021	0·356391	0·187959	2·118599	3·853363	2·437591
0·52	0·477748	0·100276	0·128840	0·379336	0·200034	2·068335$_5$	3·716608	2·333854
0·56	0·505772	0·106287	0·136507	0·401825	0·211868	2·017477$_5$	3·578121	2·228754
0·60	0·533256	0·112190	0·144032	0·423896	0·223479	1·966039	3·437943	2·122325$_5$
0·64	0·560241	0·117994	0·151427	0·445579	0·234885$_5$	1·914034	3·296113	2·014598
0·68	0·586764	0·123705	0·158701	0·466903	0·246101	1·861474	3·152667	1·905601
0·72	0·612855	0·129329	0·165861	0·487890$_5$	0·257139	1·808370	3·007637$_5$	1·795360
0·76	0·638542	0·134872	0·172916	0·508564	0·268011	1·754733	2·861056	1·683899
0·80	0·663849$_5$	0·140339	0·179871	0·528942	0·278726	1·700573	2·712952	1·577242
0·84	0·688800	0·145734	0·186732	0·549042	0·289293	1·645898	2·563351	1·457410
0·88	0·713413	0·151061	0·193505	0·568878	0·299721	1·590718	2·412280	1·342423
0·92	0·737706	0·156323	0·200193	0·588465	0·310017	1·535040	2·259762	1·226300
0·96	0·761696$_5$	0·161524	0·206801	0·607814$_5$	0·320188	1·478872	2·105820	1·109059
1·00	0·785398	0·166667	0·213333	0·626939	0·330239	1·422222	1·950476	0·990718

and

$$A_1 = \tfrac{1}{24}\left(\frac{4 - 3I_1}{\epsilon}\right). \quad (3.8)$$

Defining

$$R(X;\, \epsilon) = \left\{\frac{1 - \epsilon - X^2}{1 - X^2}\right\},$$

the required integrals

$$I_0 = \int_0^{\sqrt{(1-\epsilon)}} R^{1/2}\, \mathrm{d}X, \quad (3.9)$$

and

$$I_1 = \int_0^{\sqrt{(1-\epsilon)}} (2 - 2X^2 + \epsilon)\, R^{1/2}\, \mathrm{d}X \quad (3.10)$$

are not of the elementary type but can be readily expressed in terms of the standard elliptic functions of the first and second kinds (see Appendix A).

In Table 1 the functions $A_0(\epsilon)$ and $A_1(\epsilon)$ are tabulated correct to six decimal places at $\epsilon = 0{\cdot}00(0{\cdot}04)\, 1{\cdot}00$. Thus using expressions (3.1) and (3·6) information on the complete or partial solidification of a prism of any specified liquid may now be obtained. Results for a particular liquid are discussed later.

II—*The Tani method*

This method is a refinement of the Kármán–Pohlhausen method. We assume the two-parameter temperature distribution

$$\left.\begin{aligned}\Theta = \frac{(X^2 - 1)(Y^2 - 1)}{\epsilon}(1 - g)\\ + g\left\{\frac{(X^2 - 1)(Y^2 - 1)}{\epsilon}\right\}^2,\end{aligned}\right\} \quad (3.11)$$

which satisfies the boundary conditions (2.7a). The unknown parameters ϵ and g are then determined using the heat balance integral (3.3) and the second integral (3.4).

On substituting expression (3.11) into equations (3.3) and (3.4) we obtain, after some elementary algebra, the following equations for $\epsilon(\tau)$ and $g(\tau)$:

$$(1 - g)\frac{\mathrm{d}\tau}{\mathrm{d}\epsilon} = a_0 - a_1 g + a_2 \epsilon \frac{\mathrm{d}g}{\mathrm{d}\epsilon}, \quad (3.12)$$

and

$$\left.\begin{aligned}(J_0 + J_1 g + J_2 g^2)\epsilon \frac{\mathrm{d}g}{\mathrm{d}\epsilon}\\ = H_0 + H_1 g + H_2 g^2 + H_3 g^3.\end{aligned}\right\} \quad (3.13)$$

Here

$$\left.\begin{aligned}&J_0 = a_2 c_0 - b_0,\ J_1 = a_2 c_1 + b_1 + b_0,\\ &J_2 = a_2 c_2 - b_1,\ H_0 = d_0 - a_0 c_0,\\ &H_1 = d_1 - d_0 - a_0 c_1 + a_1 c_0,\\ &H_2 = d_2 - d_1 - a_0 c_2 + a_1 c_1 \text{ and}\\ &H_3 = a_1 c_2 - d_2;\end{aligned}\right\} \quad (3.14)$$

also

$$\left.\begin{aligned}&a_0 = \tfrac{3}{8}\beta A_0 + A_1,\ a_1 = A_1 - A_2,\\ &a_2 = A_1 - \tfrac{1}{2}A_2,\ b_0 = \tfrac{4}{3}A_2 - \tfrac{1}{3}A_3,\\ &b_1 = \tfrac{4}{3}A_2 - \tfrac{2}{3}A_3 + \tfrac{1}{2}A_4,\ c_0 = B_0,\\ &c_1 = B_1 - 2B_0,\ c_2 = B_0 - B_1 + B_2,\\ &d_0 = \beta A_0 + \tfrac{4}{3}A_2,\ d_1 = A_3 - \tfrac{8}{3}A_2 \text{ and}\\ &d_2 = \tfrac{4}{3}A_2 - A_3 + A_4.\end{aligned}\right\} \quad (3.15)$$

The functions $A_0(\epsilon)$ and $A_1(\epsilon)$ have been defined by expressions (3.7) and (3.8) respectively. The remaining functions A_r and B_r (see Table 1) are defined as follows:

$$\left.\begin{aligned}&A_2 = \frac{1}{300}\left(\frac{64 - I_2}{\epsilon^2}\right),\\ &A_3 = \frac{3}{(35)^2}\left(\frac{256 - 35I_3}{\epsilon^3}\right),\\ &A_4 = \frac{2}{(315)^2}\left(\frac{(128)^2 - 315I_4}{\epsilon^4}\right),\\ &B_0 = \frac{8}{45}\left(\frac{8 - 3S_0}{\epsilon}\right),\\ &B_1 = \frac{32}{525}\left(\frac{32 - 15S_1}{\epsilon^2}\right)\\ &\text{and}\\ &B_2 = \frac{32}{3(105)^2}\left(\frac{1024 - 105S_2}{\epsilon^3}\right).\end{aligned}\right\} \quad (3.15)$$

In Appendix A expressions, in terms of the standard elliptic functions, are obtained for the following integrals:

$$\left.\begin{aligned}
I_2 &= \int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^2 (3R^2 - 10R + 15)R^{1/2}\,\mathrm{d}X, \\
I_3 &= \int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^3 (5R^3 - 21R^2 + 35R - 35)R^{1/2}\,\mathrm{d}X, \\
I_4 &= \int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^4 (35R^4 - 180R^3 + 378R^2 - 420R + 315)R^{1/2}\,\mathrm{d}X, \\
S_0 &= 5\int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^2 R^{3/2}\,\mathrm{d}X, \\
S_1 &= \tfrac{7}{8}\int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^3 (3R^2-5)R^{3/2}\,\mathrm{d}X, \\
&\text{and} \\
S_2 &= 3\int_0^{\sqrt{(1-\epsilon)}} (X^2-1)^4 (15R^2 - 42R + 35)R^{3/2}\,\mathrm{d}X.
\end{aligned}\right\} \quad (3.17)$$

Now for the solidified phase the energy thickness Θ^*, related to the total thermal energy, is defined as

$$\Theta^* = \iint_{C_F=0}^{C_0=0} \Theta\,\mathrm{d}\sigma. \quad (3.18)$$

On using expression (3.11) we obtain

$$\Theta^* = \tfrac{32}{3}(1-g)A_1 + \tfrac{16}{3} gA_2$$

and from (B.1) it follows that

$$\Theta^* \sim (1 - \tfrac{1}{3} g)\,\epsilon \log \frac{4}{\sqrt{\epsilon}} \quad (3.19)$$

for small ϵ. Hence the first order differential equations (3.12) and (3.13) are to be solved subject to the boundary conditions

$$\tau = 0 \text{ at } \epsilon = 0 \text{ and } \lim_{\epsilon\to 0} g\epsilon \log \frac{4}{\sqrt{\epsilon}} = 0. \quad (3.20)$$

The latter condition is derived from (3.19) since $\Theta^* \to 0$ as $\epsilon \to 0$.

In Appendix B the behaviour of g is investigated when ϵ is small. Specifically we find that for $0 \leqslant \epsilon \leqslant 0{\cdot}04$ the function g is represented by the series expansion:

$$g = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{g_{p,q}\,\epsilon^q}{[\log(4/\sqrt{\epsilon})]^p}. \quad (3.21)$$

Equations are given in Appendix B from which the $g_{p,q}$ coefficients may be evaluated. Although the function g is finite and non-zero at the origin, the derivative $\mathrm{d}g/\mathrm{d}\epsilon$ is singular and behaves as $1/[\epsilon \log^2(4/\sqrt{\epsilon})]$ as $\epsilon \to 0$. Thus the differential equation (3.13) subject to (3.20) cannot be integrated numerically from the origin outwards using a finite difference step-by-step method. Briefly the procedure adopted is as follows. For a specified β the function g is evaluated for $0 \leqslant \epsilon \leqslant 0{\cdot}04$ from expression (3.21). At $\epsilon = 0{\cdot}04$ the functions g and g' are now known and we can proceed to evaluate g at 0·08(0·04) 1·0, using the method of Fox and Goodwin [6].

Once g is known, $\tau(\epsilon)$ may be found (3.12) at $\epsilon = 0{\cdot}00(0{\cdot}04)\ 1{\cdot}00$ by direct numerical integration. Thus

$$\tau = -\epsilon a_2 \log(1-g) + \int_0^{\epsilon} \left\{(a_2 + a_3)\log(1-g) + \frac{a_0 - a_1 g}{1-g}\right\}\mathrm{d}\epsilon \quad (3.22)$$

where

$$a_3 = \epsilon\frac{\mathrm{d}a_2}{\mathrm{d}\epsilon} = -a_1,$$

a result which can be verified using the information given in Appendix A.

4. RESULTS AND DISCUSSION

In Table 2 quantitative results are given, as in Ref. 1, for $\beta = 1{\cdot}5613$ evaluated using the Kármán–Pohlhausen and Tani method. The dimensionless time τ taken for the solidification front to pass through the points $X = \sqrt{(1-\epsilon)}$, $Y = 0$ or the points $X = Y = \sqrt{(1-\sqrt{\epsilon})}$ can then be deduced. These results are given graphically in Figs. 1 and 2 respectively together with the corresponding results obtained from the relaxation diagram of Allen and Severn [1]. In Figs. 1 and 2 we note that the Kármán–Pohlhausen one-parameter method and the Tani two-parameter method are in close agreement for depths of solidification corresponding to $\epsilon < \frac{1}{2}$, but for $\epsilon \sim 1$ the agreement between the two methods is not as good. Obviously the

Table 2

Uniform prism: Square cross-section					
			Approx. I	Approx. II	
ϵ	$X = (1-\epsilon)^{1/2}$, $Y = 0$	$X = Y = (1-\epsilon^{1/2})^{1/2}$	τ	τ	g
0·00	1·000	1·000	0·0000	$0{\cdot}000_0$	$-0{\cdot}087_3$
0·04	0·980	0·894	0·0010	$0{\cdot}000_9$	$-0{\cdot}083_3$
0·08	0·959	0·847	0·0037	$0{\cdot}003_4$	$-0{\cdot}068_0$
0·12	0·938	0·809	0·0077	$0{\cdot}007_2$	$-0{\cdot}051_9$
0·16	0·920	0·775	0·0131	$0{\cdot}012_4$	$-0{\cdot}035_3$
0·20	0·894	0·744	0·0197	$0{\cdot}018_8$	$-0{\cdot}017_7$
0·24	0·872	0·714	0·0274	$0{\cdot}026_5$	$+0{\cdot}000_3$
0·28	0·849	0·686	0·0362	$0{\cdot}035_5$	$0{\cdot}019_1$
0·32	0·825	0·659	0·0461	$0{\cdot}045_9$	$0{\cdot}038_5$
0·36	0·800	0·632	0·0570	$0{\cdot}057_6$	$0{\cdot}058_6$
0·40	0·775	0·607	0·0689	$0{\cdot}070_6$	$0{\cdot}079_4$
0·44	0·748	0·581	0·0818	$0{\cdot}085_2$	$0{\cdot}100_9$
0·48	0·721	0·554	0·0956	$0{\cdot}101_3$	$0{\cdot}123_2$
0·52	0·693	0·528	0·1103	$0{\cdot}119_0$	$0{\cdot}146_4$
0·56	0·663	0·502	0·1260	$0{\cdot}138_4$	$0{\cdot}170_4$
0·60	0·632	0·474	0·1425	$0{\cdot}159_6$	$0{\cdot}195_3$
0·64	0·600	0·447	0·1599	$0{\cdot}182_7$	$0{\cdot}221_3$
0·68	0·566	0·418	0·1782	$0{\cdot}208_0$	$0{\cdot}248_3$
0·72	0·529	0·389	0·1973	$0{\cdot}235_7$	$0{\cdot}276_5$
0·76	0·490	0·358	0·2172	$0{\cdot}265_9$	$0{\cdot}305_9$
0·80	0·447	0·326	0·2380	$0{\cdot}299_0$	$0{\cdot}336_7$
0·84	0·400	0·283	0·2596	$0{\cdot}335_2$	$0{\cdot}369_0$
0·88	0·346	0·249	0·2819	$0{\cdot}375_3$	$0{\cdot}402_9$
0·92	0·283	0·202	0·3051	$0{\cdot}419_6$	$0{\cdot}438_7$
0·96	0·200	0·141	0·3290	$0{\cdot}468_9$	$0{\cdot}476_6$
1·00	0·000	0·000	0·3536	$0{\cdot}524_4$	$0{\cdot}516_8$

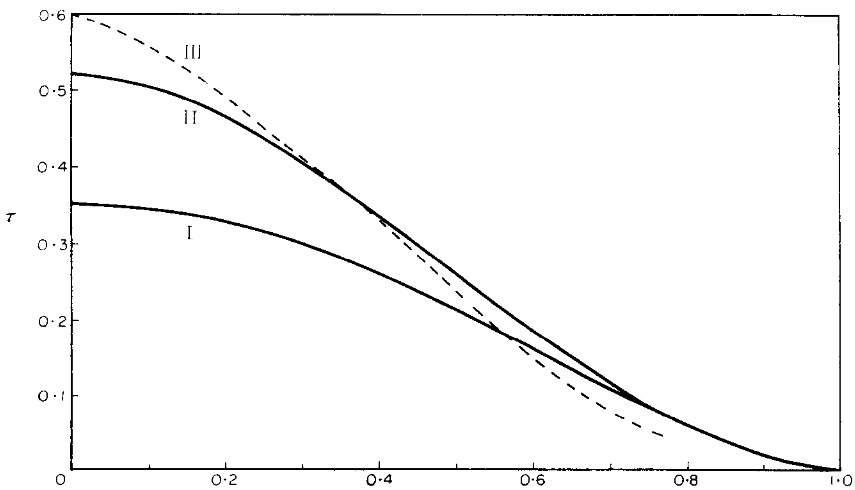

FIG. 1. Location and time history of the solidification front for a prism having square cross-section, when $X = \sqrt{(1-\epsilon)}$, $Y = 0$ and $\beta = 1{\cdot}5613$: I—Kármán–Pohlhausen method, II—Tani method and III—Allen and Severn [1].

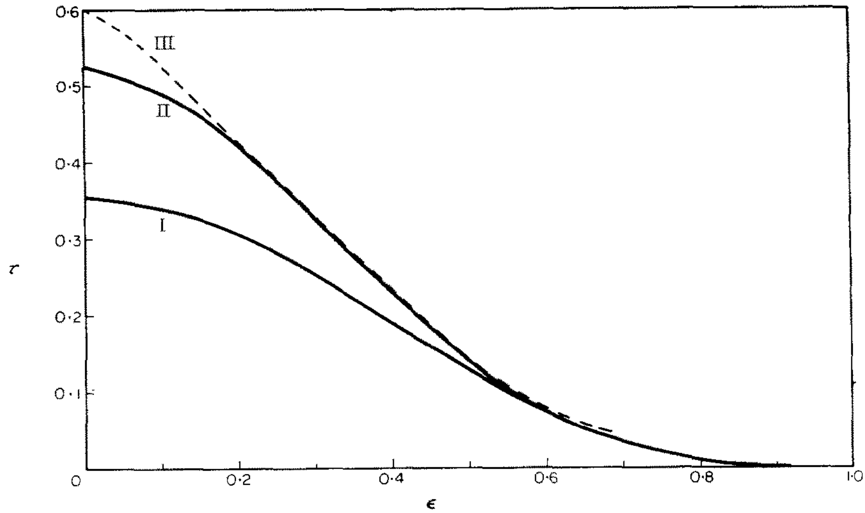

FIG. 2. Location and time history of the solidification front for a prism having square cross-section, when $X = Y = \sqrt{(1 - \sqrt{\epsilon})}$ and $\beta = 1{\cdot}5613$: I—Kármán–Pohlhausen method, II—Tani method and III—Allen and Severn [1].

Tani-method will be the most accurate for $\epsilon \sim 1$. The non-dimensional time taken for the complete solidification of the prism calculated using the Kármán–Pohlhausen and Tani methods is $\tau = 0{\cdot}35_4$ and $0{\cdot}52_4$ respectively, to be compared with the result of the relaxation solution $\tau = 0{\cdot}60$. Thus the Tani-method should be sufficient for all practical purposes, being in error by at most 13 per cent in the present calculation. If information is required for small depths of solidification the simple Kármán–Pohlhausen method should be sufficient.

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APPENDIX A

The integrals I_r and S_r can be expressed in terms of the complete elliptic integrals of the first and second kinds. In the usual notation (see Bowman [7] these integrals are defined by:

$$\left.\begin{aligned} K' &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)}\sqrt{(1-k'^2t^2)}} \\ \text{and} \qquad E' &= \int_0^1 \frac{\sqrt{(1-k'^2t^2)}}{\sqrt{(1-t^2)}}\,\mathrm{d}t \end{aligned}\right\} \quad \text{(A1)}$$

respectively. The complementary modulus k' is given by:

$$k'^2 = 1 - k^2. \quad \text{(A2a)}$$

These integrals have been calculated by Milne–Thomson [8] and in this reference a different notation for the modulus is adopted, namely

$$m_1 = k'^2 = 1 - m = 1 - k^2. \quad \text{(A2b)}$$

Consider first the evaluation of the integral (3.9), i.e.

$$\left.\begin{aligned} I_0(\epsilon) &= \int_0^{\sqrt{(1-\epsilon)}} R^{1/2}\,\mathrm{d}X, \\ \text{where} \qquad R(X;\,\epsilon) &= \left\{\frac{1-\epsilon-X^2}{1-X^2}\right\}. \end{aligned}\right\} \quad \text{(A3)}$$

Introducing the new variable

$$t = (X/\sqrt{m_1}), \quad m_1 = 1 - \epsilon,$$

the integral (A3) can, on rearrangement, be expressed in the form

$$I_0 = m_1 \left\{ \int_0^1 \frac{dt}{\sqrt{(1-m_1 t^2)}\sqrt{(1-t^2)}} - \int_0^1 \frac{t^2\, dt}{\sqrt{(1-m_1 t^2)}\sqrt{(1-t^2)}} \right\} \qquad (A4)$$

Since

$$\int_0^1 \frac{t^2\, dt}{\sqrt{(1-m_1 t^2)}(1-t^2)} = \frac{1}{m_1}(K'-E'),$$

then

$$I_0 = E' + K'(m_1 - 1) = E' - \epsilon K'. \qquad (A5)$$

On using the rules for differentiating E' and K' we obtain

$$A_0 = \tfrac{1}{2}\epsilon K'. \qquad (A6)$$

The remaining integrals I_r and S_r are evaluated using the recurrence relationship

$$\frac{d}{dv}(s^{p+1} cd) = (p+1)s^p - (p+2)(1+k'^2)s^{p+2} + (p+3)k'^2 s^{p+4}$$

where p is an even integer and s, c and d denote the Jacobian elliptic functions *snv*, *cnv* and *dnv* respectively. We obtain after some algebra, that

$$\left.\begin{aligned}
3I_1 &= (4+\epsilon)E' - \epsilon(2+3\epsilon)K',\\
I_2 &= (64+16\epsilon+9\epsilon^2)E' - \epsilon(32+12\epsilon+45\epsilon^2)K',\\
35I_3 &= (256+64\epsilon+36\epsilon^2+25\epsilon^3)E' - \epsilon(128+48\epsilon+30\epsilon^2+175\epsilon^3)K',\\
315I_4 &= (16\,384+4096\epsilon+2304\epsilon^2+1600\epsilon^3+1225\epsilon^4)E' - \epsilon(8192+3072\epsilon+1920\epsilon^2+1400\epsilon^3+11\,025\epsilon^4)K',\\
3S_0 &= (8-13\epsilon+3\epsilon^2)E' - 2\epsilon(3-\epsilon)K',\\
15S_1 &= (32+8\epsilon-83\epsilon^2+25\epsilon^3)E' - \epsilon(16+6\epsilon-40\epsilon^2)K',\\
&\text{and}\\
105S_2 &= (1024+256\epsilon+144\epsilon^2-3575\epsilon^3+1225\epsilon^4)E' - \epsilon(512+192\epsilon+120\epsilon^2-1750\epsilon^3)K'.
\end{aligned}\right\} \qquad (A7)$$

The functions $A_r(\epsilon)$ and $B_r(\epsilon)$ [see definitions (3.7), (3.8) and (3.16)] can now be evaluated numerically from (A6) and (A7) with the aid of the Milne–Thomson tables. These have been given for convenience in Table 1.

APPENDIX B

It is the purpose of this appendix to establish the behaviour of the function g, satisfying (3.13) and (3.20), when ϵ is small. Using the known expansions for K' and E' (see Bowman [7]) we obtain, on using expressions (A6) and (A7), the following expansions valid for small ϵ:

$$\left.\begin{aligned}
A_0 &= -\left\{\frac{\epsilon^2}{8} + \frac{21}{256}\epsilon^3 \ldots\right\} + \left\{\frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \frac{9\epsilon^3}{128}\ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
A_1 &= \left\{\frac{3}{128}\epsilon - \frac{5}{384}\epsilon^2 \ldots\right\} + \left\{\frac{3}{32}\epsilon + \frac{1}{64}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
A_2 &= \left\{\frac{1}{48}\epsilon - \frac{21}{1024}\epsilon^2 \ldots\right\} + \left\{\frac{1}{8}\epsilon + \frac{3}{128}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
A_3 &= \left\{\frac{3}{64}\epsilon - \frac{27}{400}\epsilon^2 \ldots\right\} + \left\{\frac{3}{8}\epsilon + \frac{3}{40}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
A_4 &= \left\{\frac{\epsilon}{50} - \frac{11}{288}\epsilon^2 \ldots\right\} + \left\{\frac{1}{5}\epsilon + \frac{1}{24}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
B_0 &= \left\{\frac{8}{3} - \epsilon \ldots\right\} + \left\{-\frac{1}{6}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},\\
B_1 &= \left\{\frac{16}{3} - \frac{8}{3}\epsilon \ldots\right\} + \left\{-\frac{1}{2}\epsilon^2 \ldots\right\}\log\frac{4}{\sqrt{\epsilon}},
\end{aligned}\right\} \qquad (B1)$$

and

$$B_2 = \left\{\frac{32}{9} - 2\epsilon \ldots\right\} + \left\{-\frac{2}{5}\epsilon^2 \ldots\right\} \log \frac{4}{\sqrt{\epsilon}}. \qquad \text{(B1)}$$

On substitution of expressions (B1) into (3.15) we may then derive, on using (3.14), corresponding expansions for the functions J_r and H_r. A careful examination of these expansions leads to the conclusion that if the J_r and H_r are to be evaluated correct to five decimals for $0 \leqslant \epsilon \leqslant 0{\cdot}04$ then terms of $0[\epsilon^3 \log^2 (4/\sqrt{\epsilon})]$ may be neglected. Thus on substitution of the J_r and H_r expressions so obtained into the differential equation (3.13) and multiplying, for convenience, both sides of this equation by $(14\,000)/(\log 4/\sqrt{\epsilon})$ we obtain for $0 \leqslant \epsilon \leqslant 0{\cdot}04$ the following simplified equation for g:

$$\left\{\left(U_0 + \frac{U_1}{\log 4/\sqrt{\epsilon}}\right) + g\left(V_0 + \frac{V_1}{\log 4/\sqrt{\epsilon}}\right) + g^2\left(W_0 + \frac{W_1}{\log 4/\sqrt{\epsilon}}\right)\right\}\epsilon \frac{dg}{d\epsilon}$$

$$= \left(L_0 + \frac{L_1}{\log 4/\sqrt{\epsilon}}\right) + g\left(M_0 + \frac{M_1}{\log 4/\sqrt{\epsilon}}\right) + g^2\left(N_0 + \frac{N_1}{\log 4/\sqrt{\epsilon}}\right) + g^3\left(R_0 + \frac{R_1}{\log 4/\sqrt{\epsilon}}\right) \qquad \text{(B2)}$$

The initial boundary condition is

$$\lim_{\epsilon \to 0} \epsilon g \log \frac{4}{\sqrt{\epsilon}} = 0.$$

The polynomial expressions U_0 and U_1 and etc., wherein the coefficients have been evaluated to the nearest whole number, are:

$$\left.\begin{aligned}
U_0 &= 600 - 390\epsilon - 88\epsilon^2 \ldots, \\
U_1 &= 325 - 224\epsilon + 40\epsilon^2 \ldots, \\
V_0 &= 840 - 180\epsilon - 69\epsilon^2 \ldots, \\
V_1 &= 269 - 216\epsilon + 27\epsilon^2 \ldots, \\
W_0 &= 160 - 130\epsilon - 31\epsilon^2 \ldots, \\
W_1 &= 73 - 77\epsilon + 13\epsilon^2 \ldots, \\
L_0 &= -1200 + (2700\beta + 1200)\epsilon \\
&\quad + (675\beta + 281)\epsilon^2 \ldots, \\
L_1 &= -500 + 444\epsilon - (675\beta \\
&\quad + 188)\epsilon^2 \ldots, \\
M_0 &= -(7200\beta + 3000) + 780\epsilon \\
&\quad + (313 - 563\beta)\epsilon^2 \ldots, \\
M_1 &= -425 + (1800\beta + 684)\epsilon \\
&\quad + (731\beta - 233)\epsilon^2 \ldots, \\
N_0 &= -(2400\beta + 1920) + (300\beta + 583) \\
&\quad - (113\beta - 166)\epsilon^2 \ldots, \\
N_1 &= -162 + (600\beta + 721)\epsilon \\
&\quad + (169\beta - 134)\epsilon^2 \ldots, \\
R_0 &= -280 + 80\epsilon + 35\epsilon^2 \ldots, \\
\text{and} \\
R_1 &= 20 + 55\epsilon - 36\epsilon^2.
\end{aligned}\right\} \text{(B3)}$$

Note that in (B.3) most of the numerical coefficients are exact.

The differential equation (B2) is satisfied by an expansion of the form

$$g = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{g_{p,\,q}\,\epsilon^q}{(\log 4/\sqrt{\epsilon})^p}. \qquad \text{(B4)}$$

On substitution of (B4) into (B2) there results a set of simultaneous equations in the coefficient $g_{p,\,q}$. The dominant coefficients may be evaluated from the following:

$$7g_{0,\,0}^3 + (60\beta + 48)g_{0,\,0}^2 + (180\beta + 75)g_{0,\,0} + 30 = 0, \qquad \text{(B5)}$$

$$\{3600 + 7200\beta + 2(2400\beta + 2340)g_{0,\,0} + 1000g_{0,\,0}^2\}g_{0,\,1}$$
$$= 2700\beta + 1200 + 780g_{0,\,0} + (300\beta + 583)g_{0,\,0}^2 + 80g_{0,\,0}^3, \qquad \text{(B6)}$$

$$\{3000 + 7200\beta + 2(2400\beta + 1920)g_{0,0} - 840g_{0,0}^2\} g_{1,0} = -500 - 425g_{0,0} - 162g_{0,0}^2 + 20g_{0,0}^3, \tag{B7}$$

and

$$\{3000 + 7200\beta + 2(2400\beta + 1920)g_{0,0} - 840g_{0,0}^2\} g_{2,0} = -425g_{1,0} - (2400\beta + 1920)g_{1,0}^2 - 324g_{0,0}g_{1,0} + 60g_{0,0}^2 g_{1,0} - 840 g_{0,0} g_{1,0}^2. \tag{B8}$$

Equations for deriving the coefficients $g_{0,2}$, $g_{0,3}$, $g_{1,1}$, $g_{1,2}$ and $g_{2,1}$ are not given as these are readily obtained. We note that for $\beta > 0$ the cubic equation (B5) has three real negative roots and we accept the smallest negative root since the other two are unacceptable on physical considerations.

Résumé—Les méthodes intégrales approchées de la théorie de la couche limite en dynamique des fluides sont utilisées pour étudier un problème de conduction thermique avec déplacement d'un front de solidification bi-dimensionnel. L'exemple traité est celui de la solidification à l'intérieur d'un prisme uniforme à section carrée rempli d'un liquide initialement à la température de fusion. Les résultats obtenus quant à la position et à l'évolution dans le temps du front de solidification sont en bon accord avec ceux trouvés par Allen et Severn [1] en utilisant la méthode de relaxation.

Zusammenfassung—Das Problem der Wärmeleitung in einer fortschreitenden, zweidimensionalen Verfestigungsfront wird mit Hilfe angenäherter Integralmethoden der hydrodynamischen Grenzschichttheorie untersucht. Als Beispiel dient die von aussen nach innen fortschreitende Verfestigung in einem Würfel, der anfänglich mit Flüssigkeit von Schmelztemperatur gefüllt war. Die Ergebnisse für die Ort- und Zeitabhängigkeit der Front sind in guter Übereinstimmung mit jenen, die Allen und Severn [1] nach der Relaxationsmethode gefunden haben.

Аннотация—Для решения задачи теплопроводности при движении плоского фронта отвердевания использованы приближённые интегральные методы теории пограничного слоя, обычные в динамике жидкостей. Рассматривается пример внутреннего отвердевания однородной призмы, имеющей квадратное поперечное сечение и заполненной жидкостью, находящейся вначале при температуре плавления. Результаты по движению фронта хорошо согласуются с данными Аллена и Северна [1], полученными ими при применении метода релаксации.